Phase in Spectral Analysis

- In the last lecture it was shown how the spectrum could be computed by 'cosine correlation' (Pd Example 14-6)
- However, this only worked because the target signals all had 'zero phase'
- The correlation between two cosines varies according to the phase difference between them
  - in-phase \(\rightarrow\) maximum correlation
  - 90° phase difference \(\rightarrow\) zero correlation
  - 180° phase difference \(\rightarrow\) maximum negative correlation
- In fact the correlation between two cosines varies as a cosine!
Phase in Spectral Analysis

\[ \cos(a) \cdot \cos(b) = \frac{1}{2} \left( \cos(a - b) + \cos(a + b) \right) \]

\[ \cos(\omega t) \cdot \cos(\omega t) = \frac{1}{2} \left( \cos(0) + \cos(2\omega t) \right) = \frac{1}{2} (1 + \cos(2\omega t)) \]

\[ \cos(\omega t) \cdot \cos(\omega t - \phi) = \frac{1}{2} \left( \cos(\phi) + \cos(2\omega t - \phi) \right) \]

\[ q = \sum_{i=0}^{N-1} \cos(\omega n T) \cdot A \cos(\omega n T - \phi) \]

\[ = \sum_{i=0}^{N-1} \left( \frac{A}{2} \left( \cos(\phi) + \cos(2\omega T - \phi) \right) \right) \]

\[ = \sum_{i=0}^{N-1} \left( \frac{A}{2} \cos(\phi) \right) + \sum_{i=0}^{N-1} \left( \frac{A}{2} \cos(2\omega T - \phi) \right) \]

\[ = \sum_{i=0}^{N-1} \left( \frac{A}{2} \cos(\phi) \right) + \sum_{i=0}^{N-1} \left( \frac{A}{2} \cos(\phi) \right) \]

\[ = \frac{AN}{2} \cos(\phi) \]

\[ = \alpha \cos(\phi) \]
Phase in Spectral Analysis

This means that cosine correlation cannot detect a $\pi/2$ ($90^\circ$) phase shift in a target signal!

Sine Correlation

The correlation between a cosine and a sine is a sine! (and it varies according to the phase difference between them)
Sine Correlation

\[ q = \sum_{n=0}^{N-1} \sin(\alpha nT), A \cos(\alpha nT - \phi) \]

\[ = \sum_{n=0}^{N-1} \cos(\alpha nT - \pi/2), A \cos(\alpha nT - \phi) \]

\[ = \sum_{n=0}^{N-1} \left( \frac{A}{2} \cos(\phi - \pi/2) + \cos(2\alpha t - \phi - \pi/2) \right) \]

\[ = \sum_{n=0}^{N-1} \left( \frac{A}{2} \sin(\phi) + \sin(2\alpha t - \phi) \right) \]

\[ = \sum_{n=0}^{N-1} \left( \frac{A}{2} \sin(\phi) \right) + \sum_{n=0}^{N-1} \left( \frac{A}{2} \sin(2\alpha t - \phi) \right) \]

\[ = \sum_{n=0}^{N-1} \left( \frac{A}{2} \sin(\phi) \right) \]

\[ = \alpha \sin(\phi) \]
Sine and Cosine Correlation

- Cosine correlation with cosine is a cosine
- Sine correlation with cosine is a sine
- Sine and cosine correlations are 90° \((\pi/2)\) out of phase
- So can cosine and sine correlation be combined in some useful way?

Amplitude and Phase

- Correlate a signal with both sines and cosines …
  
  \[
  \text{cosinecorrelation} = \alpha \cos(\phi) \\
  \text{sinecorrelation} = \alpha \sin(\phi)
  \]

- Note …
  
  \[
  (\alpha \cos(\phi))^2 + (\alpha \sin(\phi))^2 = \alpha^2
  \]

- Hence the amplitude of the sinusoidal component independent of phase is given by …
  
  \[
  \alpha = \sqrt{(\text{cosinecorrelation})^2 + (\text{sinecorrelation})^2}
  \]

- The phase of this component is given by …
  
  \[
  \tan(\phi) = \frac{\alpha \sin(\phi)}{\alpha \cos(\phi)} \\
  \Rightarrow \phi = \tan^{-1}(\frac{\text{sinecorrelation}}{\text{cosinecorrelation}})
  \]
General Spectral Analysis Algorithm

- Recall that any periodic signal can be expressed as the sum of a fundamental sinusoid and its harmonics (Fourier).
- The individual components at a frequency $\Omega = p\omega$ can be found by correlating $s(nT)$ with $\cos(\Omega nT)$ and $\sin(\Omega nT)$.
- Let $c(\Omega)$ be the cosine correlation and $s(\Omega)$ the sine correlation.

\[
c(\Omega) = \sum_{n=0}^{N-1} s(nT) \cos \left( \frac{2\pi p n}{N} \right) \quad p = 0, 1, \ldots, N-1
\]
\[
s(\Omega) = \sum_{n=0}^{N-1} s(nT) \sin \left( \frac{2\pi p n}{N} \right) \quad p = 0, 1, \ldots, N-1
\]

\[
a_p = \sqrt{c(\Omega)^2 + s(\Omega)^2}
\]
\[
\phi_p = \tan^{-1} \left( \frac{s(\Omega)}{c(\Omega)} \right)
\]

- This is the Discrete Fourier Transform (DFT).
Complex Numbers: A Reminder

- Complex numbers are represented as $z = x + jy$.
- Magnitude $|z| = \sqrt{x^2 + y^2}$.
- Phase $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
- Exponential form: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

Complex Formulation of the DFT

- The DFT is often expressed using 'complex number notation'.
- The cosine and sine correlations are associated with the real and imaginary parts of a complex number ...
Complex Formulation of the DFT

- Hence the DFT can be expressed as …

\[ S_p = \sum_{n=0}^{N-1} s(nT) e^{-j \frac{2\pi np}{N}} \quad p = 0, 1, \ldots, N - 1 \]

… where \( S_p \) is a complex number whose magnitude and phase correspond to that of the spectrum of \( s(nT) \) at a frequency \( p/NT \)

- Note also the 'inverse DFT' …

\[ s(nT) = \frac{1}{N} \sum_{p=0}^{N-1} S_p e^{j \frac{2\pi np}{N}} \quad n = 0, 1, \ldots, N - 1 \]

Using the DFT

This is the explanation for Nyquist's theorem
Time vs. Frequency Resolution

Frequency resolution = $1/NT$ (Hz)

- Increasing the analysis frame $N$ decreases the spacing between the spectral components and reduces the ability to respond to changes in the signal.
- Hence, large $N$ leads to 'narrowband analysis' with good spectral resolution and poor time resolution.

- Decreasing the analysis frame $N$ increases the spacing between the spectral components and increases the ability to respond to changes in the signal.
- Hence, small $N$ leads to 'wideband analysis' with good time resolution and poor spectral resolution.

This is the time-frequency trade-off we saw in Lecture 4.

Time vs. Frequency Resolution

In [wsprobe~] …
- narrowband analysis uses a frame/block size of 8192 samples (~190 msecs)
- wideband analysis uses a frame/block size of 512 samples (~12 msecs)

Implicit Periodicity in the DFT

- The DFT computes the spectrum at \( N \) evenly spaced discrete frequencies
- The algorithm assumes periodicity outside the analysis frame (with a period equal to the frame length)
- This means that discontinuities will arise for …
  - periodic signals with a non-integer number of cycles in the analysis frame
  - all aperiodic signals
  - all stochastic signals
- Such discontinuities give rise to unwanted spectral components
Implicit Periodicity in the DFT

- Signal as assumed by DFT.
- Analysis frame (N samples).
- Implicit periodicity.
- Spectrum of assumed signal (as derived).

Implicit Periodicity in the DFT

- Nonzero number of cycles.
- Analysis frame (N samples).
- Implicit periodicity.
- Undesirable components due to discontinuity.
Windowing

- The discontinuities arising from segmenting the signal into frames distorts the spectrum
- The distortion can be reduced by multiplying each signal frame with a ‘window function’
- The most common window function is the ‘Hamming Window’ (proposed by Richard W. Hamming) …

\[ w(nT) = 0.54 - 0.46 \cos \left( \frac{2\pi m}{N-1} \right) \]

Windowing attenuates the components caused by the discontinuity, but also smears the spectral peaks.
Windowing

- Sharper peak
- Higher floor
- Broader peak
- Lower floor

The Fast Fourier Transform (FFT)

- Implementation of the DFT requires the order of $N^2$ multiply-add operations
- By exploiting symmetry, it is possible to devise an algorithm that requires only $N \log_2 N$ multiply-add operations
  - e.g. for $N=2048$, the result is ~100x faster
- This more efficient algorithm is the … *Fast Fourier Transform* (FFT)
- The FFT requires that the window/frame should be a power of 2 in size
- This can be achieved by …
  - choosing the appropriate analysis frame size, and/or
  - zero-padding a frame to the nearest power of 2
The FFT in Pure Data

- There are two key FFT objects available in Pd …
  - [rfft~] computes the forward transform
  - [ifft~] computes the inverse transform
- We saw both [rfft~] and [ifft~] in action in Example 12-3
- The magnitude spectrum can be computed by squaring the real and imaginary outputs, then taking the square root [sqrt~]
- Alternatively, the Pd object [framp~] outputs frequencies and amplitudes directly
- [rfft~] followed by [framp~] is used in [wsprobe~]
This lecture has covered …

- Phase in spectral analysis
- Sine and cosine correlation
- The discrete Fourier transform (DFT)
- Time versus frequency resolution
- Windowing
- The fast Fourier transform (FFT)

Any Questions?
Next time …

The ‘Z’ Transform